PROBLEMS IN ALGEBRAIC SURFACES

1. Problem Set 1: Curves on Surfaces, Due July 3rd

References: Beauville section 1, Hartshorne Chapter 5, section 1.

1.1. Theorems.

The main theorems (we talked about in the lecture) that you may need include the adjunction formula and the Bezout theorem.

Theorem 1.1. *(adjunction formula) Let X be a smooth projective surface, and C a curve in X. Let K_X be the canonical divisor of X then*

$$2g(C) - 2 = C \cdot C + K_X \cdot C,$$

where $g(C)$ is the genus of the curve C (should be interpreted as arithmetic genus when C is singular).

Theorem 1.2. *(Bezout theorem) Let C_1 and C_2 be two curves in $\mathbb{P}^2$ of degree d_1 and d_2 respectively, then $C_1 \cdot C_2 = d_1 d_2$.*

1.2. Curves on $\mathbb{P}^2$.

Problem 1.3. *Let $[x : y : z]$ be the coordinate on $\mathbb{P}^2$. Find the conic (a degree two homogeneous polynomial in x, y, z) passing through the points $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$, $[1 : 1 : 1]$, and $[a : b : c]$.*

Problem 1.4. *Determine for which a, b, c the conic is reducible (i.e., become union of two lines).*

Problem 1.5. *Deduce from adjunction formula the degree-genus formula on $\mathbb{P}^2$: Any smooth curve C of $\mathbb{P}^2$ of degree d has genus $(d-1)(d-2)/2$.*

1.3. Curves on quadric surface.

Problem 1.6. *Show that the Segre embedding*

$$\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$$

$$[a : b], [c : d] \mapsto [ac : ad : bc : bd]$$

defines an isomorphism from $\mathbb{P}^1 \times \mathbb{P}^1$ to the quadric surface Q in $\mathbb{P}^3$ defined by $xw = yz$.

Problem 1.7. *Let C be a smooth curve in the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$ of type (a, b), where $a, b \geqslant 0$ are integers.*

(1) Show that $g(C) = ab - a - b + 1$.
(2) Find the genus of the curve of type $(1, 0)$, $(0, 1)$, $(1, 1)$, and $(2, 2)$.
(3) If we use the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, how are the curves above embedded in $\mathbb{P}^3$? (Remark: a (2,2) curve is usually called "Clifford torus".)

Problem 1.8. *Let $C_{1,2}$ be the curve of type $(1, 2)$ on the quadric surface.*

Date: August 2024.

(1) *Find the genus of $C_{1,2}$. Explain how is the curve of type $(1,2)$ embedded in $\mathbb{P}^3$. (There is a common name for this curve.)*
(2) *Find the normal bundle of $C_{1,2}$ in $\mathbb{P}^3$. (Hint: first think about the same question for (2,2) curve.)*

1.4. **Intersections of curves in $\mathbb{P}^2$.**

Problem 1.9. *Construct two (affine) smooth conics passing through $(0,0)$ with intersection multiplicity $1,2,3,4$, respectively.*

Problem 1.10. *Find all the intersections and the multiplicities of the two affine plane curves (working over $\mathbb{C}$)*

$$y^2 = x(x-1)(x+1), \text{ and } y^2 + x^2 - x = 0.$$

Problem 1.11. *(challenge problem)*
Find intersection multiplicity for $\frac{3}{4}y^2 + 3xy + y - x^3 = 0$ *and* $y^2 + 3xy = x^3$ *at* $(0,0)$.

1.5. **arithmetic genus.**

Problem 1.12. *Find the arithmetic genus of the curve I_3, which is the union of three rational curves, and form a loop.*

(1)

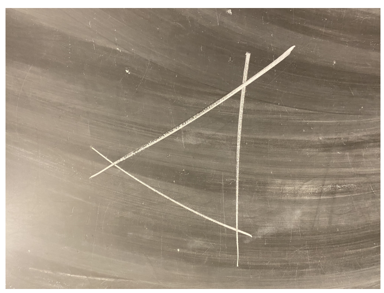

FIGURE 1. I_3

Problem 1.13. *Find the arithmetic genus of the curve I_0^*, which is a nonreduced curve with five irreducible components, and the horizontal one has multiplicity two.*

(2)

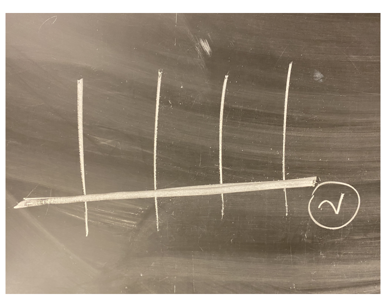

FIGURE 2. I_0^*

Remark: The notations I_3 and I_0^* come from Kodaira, and they are all singular fibers of elliptic surfaces (see Wikipedia page here). So, if you know the arithmetic genus is preserved in a flat family, you can immediately solve the two problems above. (But please don't use that as a proof here.)

2. Problem Set 2: Blow-up on Surfaces (due July 18)

2.1. **Intoduction.** In this section, we study blowup of a surface. The blow-up $\mathbb{C}^2$ at the point $(0,0)$ has the equation $(x, y; \alpha : \beta) \in \mathbb{C} \times \mathbb{P}^1$ satisfying

$$\det \begin{bmatrix} x & y \\ \alpha & \beta \end{bmatrix} = 0.$$

In affine chart $\alpha = 1$, the equation is $y = x\beta$, and in the affine chart $\beta = 1$, the equation is given by $x = y\alpha$. In the common intersection $\alpha, \beta \neq 0$, the two equations agree via $\alpha = \frac{1}{\beta}$. The exceptional divisor E is the set of all $(0, 0; \alpha : \beta)$, which is a copy of $\mathbb{P}^1$. E has self-intersection -1.

Blowup is introduced to resolve the singularities of a curve. On the other hand, one can keep track of the intersection numbers, Picard group, etc, under blowup. One refer to Beauville section II, and Hartshorne Chapter 1, section 4, Chapter 5, section 5 for details. Below are the theorems you need to solve the problems.

Theorem 2.1. *(Blow up exists) Let S be a surface and $p \in S$ a point. Then there exists a surface $\hat{S}$ and a morphism $\varepsilon : \hat{S} \to S$ which are unique up to isomorphism, such that*

(1) the restriction of ε to $\varepsilon^{-1}(S - \{p\})$ is an isomorphism onto $S - \{p\}$;
(2) $\varepsilon^{-1}(p) = E$ is isomorphic to $\mathbb{P}^1$ and called the exceptional divisor.

Now consider an irreducible curve C on S that passes through p with multiplicity m. The closure of $\varepsilon^{-1}(C - \{p\})$ in $\hat{S}$ is an irreducible curve $\hat{C}$ on $\hat{S}$, which we call the strict transform of C.

Lemma 2.2. *$\varepsilon^* C = \hat{C} + mE$.*

Theorem 2.3. *Let $\varepsilon : \hat{S} \to S$ be the blowup of a smooth surface at one point p. Then*

(1) Let D and D' be divisors on S, then $\varepsilon^* D.\varepsilon^* D' = D.D'$, $(\varepsilon^* D).E = 0$ and $E^2 = -1$.
(2) There is an isomorphism $\mathrm{Pic}(S) \oplus \mathbb{Z} \cong \mathrm{Pic}(\hat{S})$, defined by $(D, n) \mapsto \varepsilon^* D + nE$.
(3) $K_{\hat{S}} = \varepsilon^* K_S + E$.

Theorem 2.4. *(Castelnuovo contraction theorem) Let X be a smooth surface, and let $E \subseteq X$ be a smooth curve, which is isomorphic to $\mathbb{P}^1$. Suppose $E \cdot E = -1$. Then, there is a contraction morphism*

$$X \to Y$$

which sends E to a point on Y, and Y is a smooth surface.

In other words, if you have a (-1) rational curve, then it arises from the blow-up of another surface.

2.2. **Strict transform of a curve.**

Problem 2.5. *Let $X \to \mathbb{C}^2$ be the blow-up at $(0,0)$. Consider the line L_1 given by $x = 0$ and L_2 be the line given by $y = 0$. Find the strict transform (also called proper transform) of the two lines. Do they intersect?*

Problem 2.6. *Let $X \to \mathbb{C}^2$ be the blow-up at $(0,0)$. Consider the line L given by $y = 0$ and C be the conic defined by $y = x^2$. Find the strict transform of the two curves. Do they intersect? If so, what is the intersection multiplicity?*

Problem 2.7. *Let $X \to \mathbb{C}^2$ be the blow-up at $(0,0)$. Find the strict transform $\tilde{C}$ of the cuspidal curve C defined by $y^2 = x^3$.*

(1) *Is $\tilde{C}$ smooth?*
(2) *Find the intersection multiplicity of $\tilde{C}$ and E.*

Problem 2.8. *Let C_1 and C_2 be the two curves in Problem 1.11. We blow up the origin. How many points do the strict transform $\tilde{C}_1$ and $\tilde{C}_2$ intersect on the exceptional divisor? What are the intersection multiplicities?*

2.3. **Resolving a rational map.**

Problem 2.9. *(pencil of lines) Consider the rational map*

$$f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$$
$$[x : y : z] \mapsto [x : y]$$

(1) *Show that f is not defined at the point $[0, 0, 1]$.*
(2) *Let X be the blow-up of $\mathbb{P}^2$ at $[0, 0, 1]$, then show that f extends to a morphism $\tilde{f} : X \to \mathbb{P}^1$.*
(3) *Show that every fiber is a copy of $\mathbb{P}^1$.*

Problem 2.10. *(Cubic curve) Let f be the same rational map as above. Let C be the cubic curve $zy^2 = x(x + 1)(x - 1)$*

(1) *Show that C is a smooth curve. (What is its genus?)*
(2) *Show that the restriction of f to C defines a regular morphism $f|_C : C \to \mathbb{P}^1$.*
(3) *What is the degree of this map? Where are the ramification points?*

Problem 2.11. *(Pencil of conics) Recall that through 5 general points on $\mathbb{P}^2$, there is a unique conic.*

(1) *Use the results in Problem 1.3 to show that a conic passing through the four points $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$, $[1 : 1 : 1]$ have the equation*

$$Axy + Byz - (A + B)xz = 0 \tag{3}$$

(2) *We can parameterize the family by $\mathbb{P}^1$: Let C_1 be the (singular) conic defined by $B = 0$, with equation $f(x, y, z) = xy - xz = 0$ and C_2 be the conic defined by $A = 0$, with equation $g(x, y, z) = yz - xz = 0$. Then show any conic in the family* (3) *can be written as*

$$Af(x, y, z) + Bg(x, y, z) = 0, \ [A : B] \in \mathbb{P}^1. \tag{4}$$

(3) *Show that then one parameter family* (4) *(we call a pencil) defines a rational map*

$$\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$$
$$[x : y : z] \mapsto [f(x, y, z) : g(x, y, z)].$$

Find the domain of ϕ. Which points on $\mathbb{P}^2$ is the ϕ not defined?
(4) *Blow up $\mathbb{P}^2$ at the points where ϕ is not defined and call the new space X. Show that ϕ extends to a regular morphism on X.*

Problem 2.12. *(Net of conics) (cf. Beauville, p.18, Problem 3.)*

(1) *Show that a conic passing through the three points $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$ have the equation*

$$Axy + Byz + Cxz = 0, \text{ with } [A : B : C] \in \mathbb{P}^2$$

(2) *Show that the 2-dimensional family of conics above defines a rational map*

$$\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$

$$[x : y : z] \mapsto [yz : xz : xy] \tag{5}$$

(3) *Find the domain of ϕ. Which points on $\mathbb{P}^2$ is the ϕ not defined?*

(4) *Blow up $\mathbb{P}^2$ at the points where ϕ is not defined and call the new space X. Show that ϕ extends to a regular morphism on X.*

2.4. Blow Down.

Problem 2.13. *(Hartshorne, Chapter 5, problem 5.2) Let $Y \cong \mathbb{P}^1$ be a curve in a smooth algebraic surface X, and $Y \cdot Y < 0$. Show that there is a contraction $X \to X'$ which sends Y to a point.*

3. Problem Set 3: Del Pezzo Surfaces, Due Aug 1st

In this section, we study a class of rational surfaces, called del Pezzo surfaces. They are blowing up at up to 8 points in the general position.

Let $\pi : X \to \mathbb{P}^2$ be blow up at points $p_1, \dots, p_k$. The canonical bundle formula (cf. Theorem 2.3) implies $K_X = \pi^* K_{\mathbb{P}^2} + E_1 + \cdots + E_d = \pi^*\mathcal{O}(-3) + E_1 + \cdots + E_d$, where E_i is the exceptional divisor over p_i, then $-K_X$, called the anti-canonical class, is ample when $k \leqslant 8$ and $p_1, \dots, p_k$ are in general positions (3 points do not on a line, 6 points do not lie on a conic.) $h^0(X, -K_X)$ is the space of cubics on $\mathbb{P}^2$ vanishing at p_i, so it is $10-k$ dimensional (cf. problem 3.5), and the linear system $|-K_X|$ defines a morphism

$$\phi : X \to \mathbb{P}^{9-k}.$$

This is an embedding when $k \leqslant 6$.

X is called the del Pezzo surface of degree $d = 9-k$. When $d \geqslant 3$, this is an embedding.

Example 3.1. *When $k = 6$. It is blowup 6 general points on $\mathbb{P}^2$, it is a cubic surface ($d = 3$).*

Definition 3.2. *A line L on a del Pezzo surface is an irreducible curve such that $L \cdot (-K_X) = 1$ and $L^2 = -1$.*

It is well known that

Theorem 3.3. *A cubic surface has 27 lines.*

Reference: Beauville, section 4; Harthshorne, Chapter 5, section 4.

3.1. Cubics on $\mathbb{P}^2$.

Problem 3.4. *Show that the space of cubic polynomials on $\mathbb{P}^2$, i.e., $H^0(\mathbb{P}^2, \mathcal{O}(3))$ is 10 dimensional.*

Problem 3.5. *Let $p_1, \dots, p_k$ be k general points on $\mathbb{P}^2$. Show that the space of cubic polynomials on $\mathbb{P}^2$ vanishing at $p_1, \dots, p_k$ form a $10-k$ dimensional subspace.*

3.2. Blow up one or two points.

Problem 3.6. *Let X be blowup at one point on $\mathbb{P}^2$ (cf. Problem 2.9). Show that $Pic(X) \cong \mathbb{Z}^2$ and spanned by two curves E and F, where E is the exceptional curve, F is the strict transform of a line through the blowup point, and the intersection pairing is $E \cdot E = -1$, $E \cdot F = 1$ and $F \cdot F = 0$.*

Problem 3.7. *Continued from the previous problem. Let L be a line which does not pass through the blowup point. Express $L = aE + bF$ in $Pic(X)$, what is a and b? Find the intersection paring of $Pic(X)$ in terms of the new basis L and F.*

Problem 3.8. *Let X be the blowup at two points on $\mathbb{P}^2$. Find all lines on X. Find a basis of $Pic(X)$ and the intersection pairing.*

3.3. Geometry of blowing up two points $Bl_{p_1,p_2}\mathbb{P}^2$.

Problem 3.9. *Show that blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at one point is isomorphic to blow up $\mathbb{P}^2$ at two points through the following steps:*

1. *By change of coordinate we can assume we blow up $\mathbb{P}^2$ at $p = [0:1:0]$ and $q = [0:0:1]$. Denote X the blow-up surface.*

(2) *Show the proper transform $\tilde{L}$ of the the line L through the two points has self-intersection -1.*
(3) *Use Castelnuovo's theorem to show $\tilde{L}$ can be contracted, i.e., there is a birational morphsim $f : X \to Y$ such that $f(\tilde{L})$ is a point.*
(4) *It reduces to show that Y is $\mathbb{P}^1 \times \mathbb{P}^1$, or quadric surface. Recall that (cf. Problem 1.6) quadric surface has two rulings (two lines in the same ruling are disjoint, and two lines in different rulings intersect at one point). So, we need to construct two rulings on Y. They should come from two families of lines on $\mathbb{P}^2$. What are they? Try to think and figure it out by yourself without reading below.*
(5) *Let's consider the rational map*

$$\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$
$$[x : y : z] \mapsto [x : z] \times [x : y]$$

Which point is this map not defined? Show that ϕ extends to a regular morphism $\tilde{\phi}$ on X and is constant on $\tilde{L}$.
(6) *So $\tilde{\phi}$ descends to a morphism $\bar{\phi} : Y \to \mathbb{P}^1 \times \mathbb{P}^1$. Show this is an isomorphism.*

3.4. **blow up 3 points and Cremona transformations.**

Problem 3.10. *(Cremona transformations, cf. Hartshorne, p.397. Example 4.2.3)*

(1) *Show that the rational map* (5)

$$\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$
$$[x : y : z] \mapsto [yz : xz : xy]$$

is a birational involution. In other words, shows that the composition (5)

$$\phi \circ \phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$

agrees with the identity map.
(2) *Recall that in Problem 2.12 (4), we showed ϕ extends to a regular morphism $\tilde{\phi}$ on the blow up X of $\mathbb{P}^2$ at three points. Describe which curves do $\tilde{\phi}$ blow down.*
(3) *Show that ϕ extends to an isomorphism on X: In other words, show that there is a following commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & X \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \mathbb{P}^2 & \overset{\phi}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

Φ agrees with ϕ and is an isomorphism between X.
(4) *Φ is an involution (regular selfmap whose composition is identity) on X. Describe this map.*

Problem 3.11. *Let X be the blow-up of $\mathbb{P}^2$ at three points. Suppose the three points are not on the same line.*

(1) *Show that we can assume the three points to be at $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$.*
(2) *Find all (-1) lines on X.*
(3) *Find Pic(X)*
(4) *Let $H = -K_X = 3\pi^*L - E_1 - E_2 - E_3$ be the hyperplane class, find a basis of $H^\perp$ in Pic(X). Show $H^\perp$ is positive definite.*

Problem 3.12. *Show that the anticanonical divisor* $-K_X$ *of blowup of three points of* $\mathbb{P}^2$ *lying on a line is not ample by finding a curve* C *on* X *such that* $-K_X \cdot C \leqslant 0$.

3.5. **Blow up more points.**

Problem 3.13. *Let* X *be the blow-up of* $\mathbb{P}^2$ *at four points. Suppose the three points are not on the same line.*

(1) *Show that any such* X *arises from a pencil of conics on* $\mathbb{P}^2$ *(cf. Problem 2.11).*
(2) *Show that we can assume the three points to be at* $[1:0:0]$, $[0:1:0]$, $[0:0:1]$, *and* $[1:1:1]$.
(3) *Find all (-1) lines on* X.
(4) *Find* $Pic(X)$
(5) *Let* $H = -K_X = 3\pi^*L - E_1 - E_2 - E_3 - E_4$ *be the hyperplane class, find a basis of* $H^\perp$ *in* $Pic(X)$. *Show* $H^\perp$ *is positive definite.*

Problem 3.14. *(Hartshorne Chapter 5, Problem 4.13.)*

(1) *Show that blowup of 5 general points on* $\mathbb{P}^2$ *has 16 lines.*
(2) *It is a del Pezzo surface of degree 4. Show that it is the complete intersection* $Q_q \cap Q_2$ *of two quadric hypersurfaces in* $\mathbb{P}^4$.

4. Problem Set 4: ADE singularities, Due Aug 15

The reference for ADE singularities (or Du Val singularities) is Miles Reid's note:

`https://homepages.warwick.ac.uk/~masda/surf/more/DuVal.pdf`

Theorem 4.1. *(Projection formula) Let $\pi : X \to Y$ be a proper map between projective algebraic surfaces. Let C be a curve on Y and D be a curve on X. Then*

$$\pi^* C \cdot D = C \cdot \pi_* D. \tag{6}$$

Here π^* is the pullback of a divisor (cf. Theorem 2.3). $\pi_* : Div(X) \to Div(Y)$ is the pushforward, defined by $\pi_* D = d \cdot \pi(D)$ if both D and $\pi(D)$ are divisors and $D \to \pi(D)$ is d-to-1, otherwise, define $\pi_*(D) = 0$.

Note in Theorem 4.1, the surfaces do not need to be smooth, in particular, if π is the resolution of singularities, then one can use (6) to define intersection numbers of two curves on a singular surface.

4.1. Resolution of ADE singularities.

Problem 4.2. *(A_2 singularity) Resolve the singularity of the equation $x^2 + y^2 + z^3 = 0$.*

In the lecture, we saw two equations for D_4 singularities:

$$x^2 + y^3 + z^3 = 0$$
$$x^2 + y^2 z + z^3 = 0$$

Problem 4.3. *Show the two equations are analytically equivalent. (Hint: you can view the surface as the double cover of $\mathbb{C}^2$ defined by $y^3 + z^3 = 0$ or $y^2 z + z^3 = 0$. Find the zero loci of two affine curves.)*

Problem 4.4. *(Reid, Exercise 2) Do the resolution for $x^2 + y^3 + z^3 = 0$. (Hint: This is a simple exercise in not missing a singularity "at infinity", which will happen if you only take the obvious coordinate piece of the blowup.) Compare with the resolution did by Mert for the other equation.*

4.2. Quotient singularity.

Problem 4.5. *(quotient singularity) Consider the $\mathbb{Z}_2$ group action on the affine plane $\tau : \mathbb{C}^2 \to \mathbb{C}^2$, $(x, y) \mapsto (-x, -y)$.*

1. *Show that the invariant subring $\mathbb{C}[x, y]^{\mathbb{Z}_2}$ of $\mathbb{C}[x, y]$ is generated by u^2, uv, and v^2.*
2. *Let $X = Spec(\mathbb{C}[x, y]^{\mathbb{Z}_2})$. Let $\phi : \mathbb{A}^2 \to X$ be the quotient defined by $x = u^2, y = uv, z = v^2$. Show that X has equation $xz = y^2$, and has an A_1 isngularity.*

Problem 4.6. *Show that the action τ extends to the blowup $\tilde{\tau} : Bl_{(0,0)}\mathbb{A}^2 \to Bl_{(0,0)}\mathbb{A}^2$. What the action does on the exceptional divisor?*

Problem 4.7. *(Reid, Exercise 1) Show that there is a commutative diagram*

$$\begin{tikzcd} Bl_{(0,0)}\mathbb{A}^2 \arrow[r, "\tilde{\phi}"] \arrow[d, "\sigma"] & \tilde{X} \arrow[d, "\sigma'"] \\ \mathbb{A}^2 \arrow[r, "\phi"] & X \end{tikzcd}$$

where

- *σ is the blowup map, ϕ is the $\mathbb{Z}_2$ quotient as before*
- *σ' is the minimal resolution of A_1 singularity, with exceptional curve E*
- *$\tilde{\phi}$ is branched double cover of $\tilde{X}$ along E.*

Problem 4.8. *Show that $E \cdot E = -2$ using projection formula for $\tilde{\phi}$.*

4.3. **Projection formula.** Let X be an algebraic surface an isolated singularity of ADE type. Let $\pi : \tilde{X} \to X$ be the minimal resolution. Let E be the exceptional divisor (so $E = \cup E_i$ is the union of rational curves as the Dynkin diagram). Recall that in the lecture, we showed that $E \cdot E = -2$ using adjunction formula and the fact that π is crepant.

Problem 4.9. *Show that each irreducible component E_i has self-intersection -2 using projection formula* (6).